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The Homotopy Type of Complexes of Graph Homomorphisms between Cycles*

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Abstract. In this paper we study the homotopy type of $\mathtt{Hom}(C_m, C_n)$, where C_k is the cyclic graph with k vertices. We enumerate connected components of $\mathtt{Hom}(C_m, C_n)$ and show that each such component is either homeomorphic to a point or homotopy equivalent to S^1. Moreover, we prove that $\mathtt{Hom}(C_m, L_n)$ is either empty or is homotopy equivalent to the union of two points, where L_n is an n-string, i.e., a tree with n vertices and no branching points.

1. Introduction

To any two graphs T and G one can associate a cell complex $\mathtt{Hom}(T, G)$, see Definition 2.2. The motivation for considering $\mathtt{Hom}(T, G)$ came from the fact that it has good structural properties, and that some special cases yield previously known constructions. For example, $\mathtt{Hom}(K_2, G)$ is homotopy equivalent to the neighborhood complex $\mathcal{N}(G)$, which plays the central role in the Lovász proof of the Kneser Conjecture in 1978, see [9].

On the other hand, since Babson and Kozlov proved the Lovász Conjecture [3] stating that for any graph G, and $r \geq 1$, $k \geq -1$:

$$\textit{if } \mathtt{Hom}(C_{2r+1}, G) \textit{ is } k\textit{-connected, then } \chi(G) \geq k+4,$$

it has become increasingly clear that the topology of $\mathtt{Hom}$-complexes carries vital information pertaining to obstructions to the existence of graph colorings. We refer the reader to the survey article [8] for an introduction and further facts about $\mathtt{Hom}$-complexes.

Until now, the homotopy type of $\mathtt{Hom}(T, G)$ was computed only in a very few special cases. It was proved in [2] that $\mathtt{Hom}(K_m, K_n)$ is homotopy equivalent to a wedge of

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$(n-m)$-dimensional spheres. It was also shown that "folding" (i.e., removing a vertex v such that there exists another vertex u whose set of neighbors contains that of v) the graph T does not change the homotopy type of $\mathtt{Hom}(T, G)$. This means, for example, that $\mathtt{Hom}(T, K_n) \simeq S^{n-2}$, where T is a tree, since one can fold any tree to an edge, and since, as also shown in [2], $\mathtt{Hom}(K_2, K_n)$ is homeomorphic to S^{n-2}. Beyond these, and a few other either degenerate or small examples, nothing is known.

In this paper we study the homotopy type of the complex of graph homomorphisms between two cycles, $\mathtt{Hom}(C_m, C_n)$, in particular $\mathtt{Hom}(C_m, K_3)$, as well as between a cycle and a string. It is easy to see that the connected components of $\mathtt{Hom}(C_m, C_n)$ can be indexed by the signed number of times C_m wraps around C_n (with an additional parity condition if m and n are even). Also, if n divides m, and C_m wraps around C_n m/n times in either direction, then, since there is no freedom to move, the corresponding connected components are points. It was further noticed by the authors, that up to now in all computed cases of $\mathtt{Hom}(C_m, C_n)$ it turned out that all other connected components were homotopy equivalent to S^1. The main result of this paper, Theorem 6.1, states that this is the case in general.

Our proof combines the methods of Discrete Morse Theory with the classical homotopy gluing construction [6, Section 4.G]. In order to be able to phrase our combinatorial argument concisely, we develop a new encoding system for the cells of $\mathtt{Hom}(C_m, C_n)$. Namely, we index the cells with collections of marked points and pairs of points on circles of length m, and translate the boundary relation into this language.

The case $n = 4$ is a bit special and is dealt with separately, using the fact that the folds are allowed in the second argument of $\mathtt{Hom}(-, -)$ as well, as long as the removed vertex has an exact double in the set of the remaining vertices, see Lemma 3.1. The homotopy type of the complex $\mathtt{Hom}(C_m, L_n)$ is computed by a similar argument.

2. Basic Notations and Definitions

For any graph G, we denote the set of its vertices by $V(G)$, and the set of its edges by $E(G)$, where $E(G) \subseteq V(G) \times V(G)$. In this paper we consider only undirected graphs, so $(x, y) \in E(G)$ implies that $(y, x) \in E(G)$. Also, our graphs are finite and may contain loops.

- For a natural number k we introduce the following notation $[k] = \{1, 2, \dots, k\}$.
- For a graph G and $S \subseteq V(G)$ we denote by $G[S]$ the graph on the vertex set S induced by G, that is $V(G[S]) = S$, $E(G[S]) = (S \times S) \cap E(G)$. We denote the graph $G[V(G) \setminus S]$ by $G - S$.
- Let $\mathtt{N}(v)$ be the set of all neighbors of $v \in V(G)$, for a graph G, that is the set $\{w \in V(G) \mid (v, w) \in E(G)\}$.
- For an integer $n \geq 2$, denote with C_n and L_n, graphs such that $V(C_n) = V(L_n) = [n]$ and $E(C_n) = \{(x, x+1), (x+1, x) \mid x \in \mathbb{Z}_n\}$, $E(L_n) = \{(x, x+1), (x+1, x) \mid x \in [n-1]\}$.

Definition 2.1. For two graphs G and H, a **graph homomorphism** from G to H is a map $\phi\colon V(G) \to V(H)$, such that if $x, y \in V(G)$ are connected by an edge, then $\phi(x)$ and $\phi(y)$ are also connected by an edge.

We denote the set of all homomorphisms from G to H by $\mathtt{Hom}_0(G, H)$.

The next definition is due to Lovász and was stated in this form in [1].

Definition 2.2. $\mathtt{Hom}(G, H)$ is a polyhedral complex whose cells are indexed by all functions $\eta\colon V(G) \to 2^{V(H)}\setminus\{\varnothing\}$, such that if $(x, y) \in E(G)$, then for all $\tilde{x} \in \eta(x)$ and $\tilde{y} \in \eta(y)$, $(\tilde{x}, \tilde{y}) \in E(H)$.

The closure of a cell η, $\mathrm{Cl}(\eta)$, consists of all cells indexed by $\tilde{\eta}\colon V(G) \to 2^{V(H)}\setminus\{\varnothing\}$ which satisfy the condition that $\tilde{\eta}(v) \subseteq \eta(v)$, for all $v \in V(G)$.

It is easy to see that the set of vertices of $\mathtt{Hom}(G, H)$is $\mathtt{Hom}_0(G, H)$. Cells of $\mathtt{Hom}(G, H)$are direct products of simplices and the dimension of a cell η is equal to $\sum_{v\in V(G)} |\eta(x)| - |V(G)|$.

Definition 2.3. For an integer i, let $[i]_m$ be an integer such that $[i]_m \in [m]$ and $i \equiv [i]_m (\text{mod } m)$.

In this paper we deal mostly with $\mathtt{Hom}(C_m, C_n)$. In this case each vertex is denoted with m-tuple $(a_1, a_2, \ldots, a_m)$, such that $a_i \in [n]$ and $[a_{[i+1]_m} - a_i]_n \in \{1, n-1\}$, for all $i \in [m]$. We also see that all cells of these complexes are cubes, since they are direct products of simplices and, clearly, the dimension of each simplex in this product is either 1 or 0.

Some Examples of $\mathtt{Hom}(C_m, C_n)$Complexes:

- $\mathtt{Hom}(C_m, C_n)$is an empty set if m is odd and n is even: if n is even, then there exists a map $\varphi\colon C_n \to K_2$, so if $\psi\colon C_m \to C_n$ exists then $\varphi \circ \psi\colon C_m \to K_2$ implying that $\chi(C_m) \leq 2$.
- Examples when $n = 3$ and $m = 2, 4, 5, 6, 7$ can be found in [2]. The number of connected components of $\mathtt{Hom}(C_m, C_3)$ was computed there and it was proven that each connected component is either a point or homotopy equivalent to S^1. The following three cases were also observed in [2].
- $\mathtt{Hom}(C_{2r+1}, C_{2q+1}) = \emptyset$ if and only if $r < q$.
- $\mathtt{Hom}(C_{2r+1}, C_{2r+1})$ is a disjoint union of 4r+2 points, for $r \geq 1$.
- $\mathtt{Hom}(C_{2r+1}, C_{2r-1})$ is a disjoint union of two cycles, the length of each of them equal to $4r^2 - 1$.
- $\mathtt{Hom}(C_4, C_{2r+1})$, for $r \geq 1$, is connected and it has $4r + 2$ squares linked in the way depicted in Fig. 1.
- $\mathtt{Hom}(C_4, C_{2r})$, for $r > 2$, has two isomorphic connected components, each of them has $2r$ squares (see Fig. 1).
- $\mathtt{Hom}(C_9, C_3)$ consists of six isolated points and two additional isomorphic connected parts, each of them has 90 solid cubes, 27 squares, 567 edges and 252 vertices. The local structure of one of those parts is shown in Fig. 2. The length of the cycle which is bold in the picture is 27.

3. Complex $\mathtt{Hom}(C_{2m}, C_4)$

We discuss this case separately because, unlike other cycles, C_4 has two vertices u and v such that $\mathtt{N}(u) = \mathtt{N}(v)$. In this section we first prove a lemma for general $\mathtt{Hom}$ complexes, and then apply it to decide the homotopy type of $\mathtt{Hom}(C_{2m}, C_4)$.

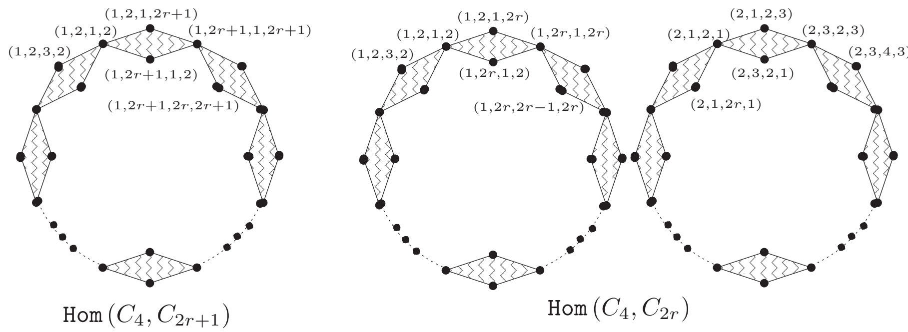

Fig. 1

Lemma 3.1. *Let G and H be graphs and let $u, v \in V(H)$ such that $\mathtt{N}(u) = \mathtt{N}(v)$. Also, let $i\colon H - v \hookrightarrow H$ be the inclusion and let $\omega\colon H \to H - v$ be the unique graph homomorphism which maps v to u and fixes other vertices. Then these two maps induce homotopy equivalences $i_H\colon \mathtt{Hom}(G, H - v) \to \mathtt{Hom}(G, H)$ and $\omega_H\colon \mathtt{Hom}(G, H) \to \mathtt{Hom}(G, H - v)$, respectively.*

Remark. A similar theorem about the reduction of certain Hom complexes was proven in [2, Proposition 5.1]. This lemma was also proven independently in [4].

Proof. We will show that ω_H satisfies conditions (A) and (B) of Proposition 3.2 from [2]. Unfolding definitions, we see that for a cell of $\mathtt{Hom}(G, H)$, $\tau\colon V(G) \to 2^{V(H)}\setminus\{\emptyset\}$, we have

$$\omega_H(\tau)(x) = \begin{cases} \tau(x), & \text{if} \quad v \notin \tau(x); \\ (\tau(x) \cup \{u\})\setminus\{v\}, & \text{otherwise.} \end{cases}$$

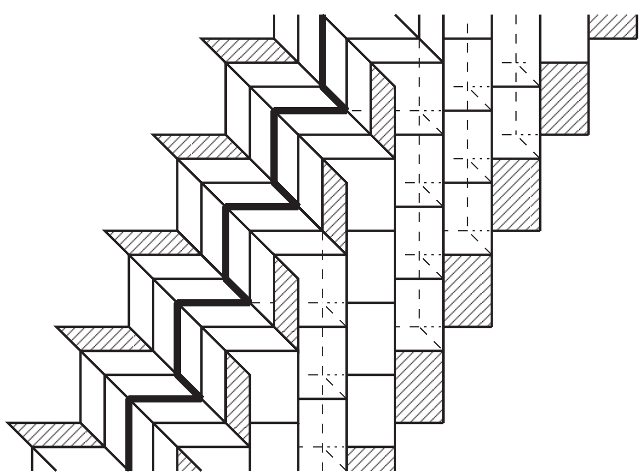

Fig. 2

Let η be a cell of $\texttt{Hom}(G, H - v)$, $\eta\colon V(G) \to 2^{V(H)\setminus\{v\}}\setminus\{\emptyset\}$. Then $\mathcal{P}(\omega_H)^{-1}(\eta)$ is the set of all η' such that, for all $x \in V(G)$,

$$\begin{cases} \eta'(x) = \eta(x), & \text{if} \quad u \notin \eta(x); \\ \eta'(x) \cap \{u, v\} \neq \emptyset \quad \text{and} \quad \eta'(x)\setminus\{u, v\} = \eta(x)\setminus\{u\}, & \text{otherwise.} \end{cases} \tag{$*$}$$

It is easy to see that, because of the condition $\texttt{N}(u) = \texttt{N}(v)$, all η' satisfying $(*)$ belong to $\texttt{Hom}(G, H)$. Take $\zeta \in \mathcal{P}(\omega_H)^{-1}(\eta)$ such that

$$\zeta(x) = \begin{cases} \eta(x), & u \notin \eta(x), \\ \eta(x) \cup \{v\}, & u \in \eta(x), \end{cases} \qquad \text{for all} \quad x \in V(G).$$

Obviously, ζ is the maximal element of $\mathcal{P}(\omega_H)^{-1}(\eta)$. It follows that $\Delta(\mathcal{P}(\omega_H)^{-1}(\eta))$ is contractible and condition (A) is satisfied.

Take now any $\tau \in \mathcal{P}(\omega_H)^{-1}(\texttt{Hom}(G, H - v)_{\geq\eta})$. Then $\eta(x)\setminus\{u\} \subseteq \tau(x)\setminus\{u, v\}$ for all $x \in V(G)$ and, if $u \in \eta(x)$, then $\tau(x) \cap \{u, v\} \neq \emptyset$. The set $\mathcal{P}(\omega_H)^{-1}(\eta) \cap \mathcal{P}(\texttt{Hom}(G, H))_{\leq\tau}$ consists of all cells η' such that, for $x \in V(G)$,

$$\begin{cases} \eta'(x) = \eta(x), & \text{if} \quad u \notin \eta(x); \\ \eta'(x) \cap \{u, v\} \neq \emptyset \quad \text{and} \quad \eta(x)\setminus\{u\} \subseteq \eta'(x) \subseteq (\eta(x) \cup \{v\}) \cap \tau(x), & \text{otherwise,} \end{cases}$$

and hence has the maximal element ξ, where $\xi(x) = \eta(x)$ for $x \in V(G)$ such that $u \notin \eta(x)$ and $\xi(x) = \tau(x) \cap (\eta(x) \cup \{v\})$ otherwise.

Since it satisfies conditions (A) and (B), we conclude that $\mathrm{Bd}(\omega_H)$ and hence also ω_H are homotopy equivalences.

It is left to prove that i_H is also a homotopy equivalence. It is clear that $\omega_H \circ i_H = id_{\texttt{Hom}(G,H-v)}$. Let ϑ be the homotopy inverse of ω_H. Then we have $i_H \circ \omega_H \simeq \vartheta \circ \omega_H \circ i_H \circ \omega_H \simeq \vartheta \circ \omega_H \simeq id_{\texttt{Hom}(G,H)}$. □

Now we have everything we need to prove the following theorem.

Theorem 3.2. *The complex* $\texttt{Hom}(C_{2m}, C_4)$, *for* $m \geq 1$, *is homotopy equivalent to a complex consisting of two points.*

Proof. We use Lemma 3.1 and obtain

$$\texttt{Hom}(C_{2m}, C_4) \simeq \texttt{Hom}(C_{2m}, L_3) \simeq \texttt{Hom}(C_{2m}, L_2).$$

It is trivial to see that $\texttt{Hom}(C_{2m}, L_2)$ has two vertices, namely $(1, 2, 1, 2, \ldots, 1, 2)$ and $(2, 1, 2, 1, \ldots, 2, 1)$, and no other cells. □

4. Discrete Morse Theory

In this section we introduce the notations and state the reformulation of Forman's result from Discrete Morse Theory given in [2]. For more general results about this topic, see [5].

Definition 4.1. A *partial matching* on a poset P with covering relation $\succ$ is a set $S \subseteq P$ together with an injective map $\mu\colon\ S \to P\backslash S$ such that $\mu(x) \succ x$, for all $x \in S$. The elements from $P\backslash(S \cup \mu(S))$ are called *critical*.

Definition 4.2. A matching is called *acyclic* if there does not exist a sequence $x_0, x_1, \dots, x_t = x_0 \in S$ such that $x_0 \neq x_1$ and $\mu(x_i) \succ x_{i+1}$, for $i \in [t-1]$.

For a regular CW complex X let $\mathcal{P}(X)$ be its face poset with covering relation $\succ$.

Proposition 4.3. *Let X be a regular CW complex, and let X' be a subcomplex of X. Then the following are equivalent*:

(1) *there is a sequence of collapses leading from X to X'*;
(2) *there is an acyclic partial matching μ on $\mathcal{P}(X)$ with the set of critical cells being $\mathcal{P}(X')$.*

For a proof see Proposition 5.4 of [7].

5. Another Notation for the Cells of $\mathtt{Hom}(C_m, C_n)$

Remark. From now on, unless otherwise stated, we work only with $\mathtt{Hom}(C_m, C_n)$ where $n \neq 4$.

Definition 5.1. We say that $i \in [m]$ is *a returning point* of a vertex $(a_1, a_2, \dots, a_m)$ of $\mathtt{Hom}(C_m, C_n)$ if $[a_i - a_{[i+1]_m}]_n = 1$.

We see that each vertex $(a_1, a_2, \dots, a_m) \in \mathtt{Hom}_0(C_m, C_n)$ uniquely determines an $(r+1)$-tuple $(i; i_1, \dots, i_r)$, where $m = nk+2r$, $i = a_1$ and $i_1, \dots, i_r$ are all its returning points with the condition that $i_1 < i_2 < \dots < i_r$. Conversely, assume that we have a $(\rho+1)$-tuple $(j; j_1, \dots, j_\rho)$, where $0 \le \rho \le m$, $j \in [n]$, $1 \le j_1 < j_2 < \dots < j_\rho \le m$ and $m = kn + 2\rho$, for some integer k. Then $(a_1, a_2, \dots, a_m)$, defined by $a_i = [j + i - 1 - 2\rho_i]_n$, where $\rho_i = |\{q \mid j_q < i\}|$, is a vertex of $\mathtt{Hom}(C_m, C_n)$with returning points $j_1, \dots, j_\rho$. Indeed, $[a_m - a_1]_n = [j + m - 2\rho \pm 1 - j]_n = [kn \pm 1]_n \in \{1, n-1\}$, where $[a_m - a_1]_n = 1$ if and only if $m = j_\rho$; and for all $i \in [m-1]$, $[a_i - a_{i+1}]_n = 1$ if and only if $i = j_q$ for some $q \in [\rho]$, otherwise $[a_i - a_{i+1}]_n = n-1$. Hence, we have proven the following lemma:

Lemma 5.2. *Let S be a set containing all $(r+1)$-tuples $(i; i_1, \dots, i_r)$, such that $0 \le r \le m$ and, for some integer k, $m = nk + 2r$, $i \in [n]$ and $1 \le i_1 < \dots < i_r \le m$. Then there is a bijection Ξ between $\mathtt{Hom}_0(C_m, C_n)$ and S given by*

$$\Xi((a_1, a_2, \dots, a_m)) = (i; i_1, \dots, i_r),$$

where $i = a_1$ and $i_1, \dots, i_r$ are all returning points of $(a_1, a_2, \dots, a_m)$.

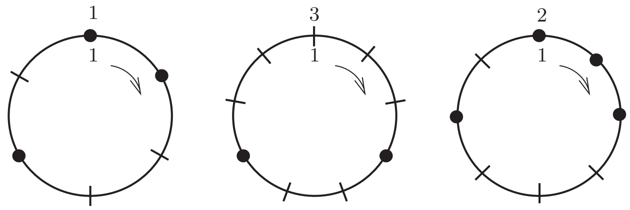

Fig. 3

Sometimes we represent a vertex $(i; i_1, \ldots, i_r)$ of $\mathtt{Hom}(C_m, C_n)$by a picture of C_m with emphasized returning points and number i. Some examples of such representation are shown in Fig. 3. Vertices of C_m are always ordered like this: if we start from the vertex labeled with 1 and go in the clockwise direction, we get an increasing sequence of numbers from 1 to m.

Lemma 5.3. *If two vertices are in the same connected component of* $\mathtt{Hom}(C_m, C_n)$, *then their number of returning points is the same.*

Proof. It is enough to prove that for an arbitrary edge, its endpoints have the same number of returning points.

Let $(a_1, \ldots, a_i, \{[a_i - 1]_n, [a_i + 1]_n\}, a_i, a_{i+3}, \ldots, a_m)$ be an edge, and let x and y be its endpoints, $x = (a_1, \ldots, a_i, [a_i - 1]_n, a_i, a_{i+3}, \ldots, a_m)$ and $y = (a_1, \ldots, a_i, [a_i + 1]_n, a_i, a_{i+3}, \ldots, a_m)$. Now it is trivial to see that, for $j \in [m]\backslash\{i, i+1\}$, j is a returning point for x if and only if it is a returning point for y. Also, we see that i is a returning point for x, while $i + 1$ is not and, similarly, $i + 1$ is a returning point for y and i is not. Hence, x and y have the same number of returning points. $\square$

It follows from definitions that each cell $\eta \in \mathtt{Hom}(C_m, C_n)$ has the property that for any $x \in [m]$, $|\eta(x)| \in \{1, 2\}$ and $\eta(x)$, $\eta([x + 1]_m)$ cannot both have cardinality 2. Also, if $|\eta(x)| = 2$, then for some $i \in [n]$, $\eta([x - 1]_m) = \eta([x + 1]_m) = \{i\}$ and $\eta(x) = \{[i - 1]_n, [i + 1]_n\}$.

Now, let $\eta \in \mathtt{Hom}(C_m, C_n)$ be a cell. Since $\mathrm{Cl}(\eta)$ is connected, by Lemma 5.3, all its vertices have the same number of returning points. Denote that number with r. Then we can denote the cell η with $(r+1)$-tuple of symbols $(s; s_{i_1}, \ldots, s_{i_r})$, where for all $j \in [r]$, $i_j \in [m]$, $1 \leq i_1 < \cdots < i_r \leq m$ and

$$s_{i_k} = \begin{cases} i_k^+, & \text{if} \quad |\eta([i_k + 1]_m)| = 2; \\ i_k, & \text{if} \quad \eta(i_k) = \{j\} \quad \text{and} \quad \eta([i_k + 1]_m) = \{[j - 1]_n\} \quad \text{for some} \quad j \in [n]. \end{cases}$$

Also, if $\eta(1) = \{i, [i + 2]_n\}$ or if $\eta(1) = \{i\}$, for some $i \in [n]$, then $s = i$. Note that, if $s_{i_k} = i_k^+$, then $[i_k + 1]_m \neq i_{[k+1]_r}$.

It is clear that each cell uniquely determines such an $(r + 1)$-tuple of symbols and that the dimension of a cell is equal to the number of s_{i_k} such that $s_{i_k} = i_k^+$.

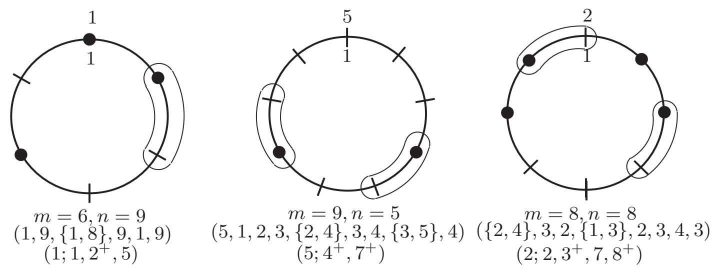

Fig. 4

Conversely, if $(j; s_{j_1}, \ldots, s_{j_r})$ is an $(r+1)$-tuple of symbols and if the following conditions are satisfied:

1. $0 \leq r \leq m$, $j \in [n]$, $m = nk + 2r$ and $1 \leq j_1 < j_2 < \cdots < j_r \leq m$,
2. for all $k \in [r]$, $s_{j_k} \in \{j_k, j_k^+\}$,
3. if $s_{j_k} = j_k^+$ then $[j_k + 1]_m \neq j_{[k+1]_r}$,

then it is not hard to check that $(j; s_{j_1}, \ldots, s_{j_r})$ corresponds exactly to one cell from $\mathtt{Hom}(C_m, C_n)$, namely to $(A_1, \ldots, A_m)$, where for $a_k = j + k - 1 - 2|\{q \mid j_q < k\}|$,

$$A_k = \begin{cases} \{[a_k]_n\}, & \text{if} \quad [k-1]_m \notin \{i_l \mid s_{i_l} = i_l^+\}; \\ \{[a_k]_n, [a_k + 2]_n\}, & \text{otherwise.} \end{cases}$$

We also represent those $(r+1)$-tuples (and corresponding cells) with pictures (see Fig. 4).

Remark. Assume $\eta = (s; s_{i_1}, \ldots, s_{i_r}) \in \mathtt{Hom}(C_m, C_n)$, and $s_{i_k} = i_k^+$, for $i_k \neq m$. "Unplusing" η in the kth position yields two $(\dim(\eta) - 1)$-dimensional cells in $\mathrm{Cl}(\eta)$: $(s; s_{i_1}, \ldots, s_{i_{k-1}}, i_k, s_{i_{k+1}}, \ldots, s_{i_r})$ and $(s; s_{i_1}, \ldots, s_{i_{k-1}}, i_k + 1, s_{i_{k+1}}, \ldots, s_{i_r})$.

Lemma 5.4. *Let $\eta = (i; s_{i_1}, \ldots, s_{i_r})$ be a cell from $\mathtt{Hom}(C_m, C_n)$ of dimension $d \geq 1$ and let $D = \{k \mid k \in [r]$ and $s_{i_k} = i_k^+\}$. For $k \in D \setminus \{m\}$, let*

$$\begin{aligned} \eta_k &= (i; s_{i_1}, \ldots, s_{i_{k-1}}, i_k, s_{i_{k+1}}, \ldots, s_{i_r}), \\ \eta'_k &= (i; s_{i_1}, \ldots, s_{i_{k-1}}, i_k + 1, s_{i_{k+1}}, \ldots, s_{i_r}). \end{aligned}$$

Then the set of all cells of $\mathtt{Hom}(C_m, C_n)$ which are contained in $\mathrm{Cl}(\eta)$ and have dimension $d-1$ is equal to:

(1) $\bigcup_{k \in D} \{\eta_k, \eta'_k\}$, *if $m \notin D$.*
(2) $\bigcup_{k \in D, k \neq m} \{\eta_k, \eta'_k\} \cup \{(i; s_{i_1}, \ldots, s_{i_{r-1}}, m), ([i+2]_n; 1, s_{i_1}, \ldots, s_{i_{r-1}})\}$, *if $m \in D$.*

Proof. If we write the cell η using old notation, it is clear that $\mathrm{Cl}(\eta)$ contains exactly $2d$ cells of dimension $d-1$. By the previous remark, we know that $S = \bigcup_{k \in D \setminus \{m\}} \{\eta_k, \eta'_k\}$ is a set consisting of different $(d-1)$-dimensional cells of $\mathrm{Cl}(\eta)$.

In case (1), $|S| = 2d$ and S is exactly the set of all $(d-1)$-dimensional cells in $\mathrm{Cl}(\eta)$.

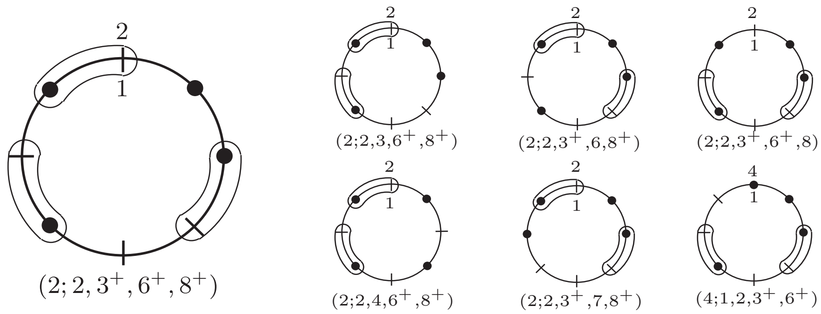

Fig. 5. A cell from $\text{Hom}(C_8, C_8)$ of dimension 3 and all 2-dimensional cells contained in its closure.

We now deal with case (2). Then $\eta = (\{i, [i+2]_n\}, [i+1]_n, \dots, [i+1]_n)$. Two $(d-1)$-dimensional cells in $\text{Cl}(\eta)$ are $\eta' = (i, [i+1]_n, \dots, [i+1]_n)$ and $\eta'' = ([i+2]_n, [i+1]_n, \dots, [i+1]_n)$ (we have changed only the first component). It is easy to see that, in new notation, $\eta' = (i; s_{i_1}, \dots, s_{i_{r-1}}, m)$ and $\eta'' = ([i+2]_n; 1, s_{i_1}, \dots, s_{i_{r-1}})\}$. The claim follows since $|S \cup \{\eta', \eta''\}| = 2d$ and all cells are different. $\square$

Remark. From our definition of $\text{Cl}(\eta)$ we see that $\text{Cl}(\eta)\backslash\{\eta\} = \bigcup_{\eta' \in S} \text{Cl}(\eta')$, where S is the set of all cells of dimension $\dim(\eta) - 1$ contained in $\text{Cl}(\eta)$.

Lemma 5.5. *Suppose $[r]_m \neq m$. The two vertices $(i; i_1, \dots, i_r)$ and $(j; j_1, \dots, j_r)$ of $\text{Hom}(C_m, C_n)$ are in the same connected component if and only if $[i+2l]_n = j$, for some integer l.*

Proof. Suppose first that $[i+2l]_n = j$.

By the previous remark, if $k \neq r, i_{k+1} > i_k + 1$ or $k = r, i_k \neq m$, then vertices $(i; i_1, \dots, i_r)$ and $(i; i_1, \dots, i_{k-1}, i_k + 1, i_{k+1}, \dots, i_r)$ are in the same connected component (both of them are elements of $\text{Cl}(i; i_1, \dots, i_{k-1}, i_k^+, i_{k+1}, \dots, i_r)$). Let us introduce an equivalence relation $\sim$ on the set $\text{Hom}_0(C_m, C_n)$ as follows: $x \sim y$ if and only if x and y lie in the same connected component. Using these two things, it is easy to see that

$$\begin{aligned}(i; i_1, \dots, i_{r-1}, i_r) &\sim (i; i_1, \dots, i_{r-1}, m) \sim (i; i_1, \dots, m-1, m) \sim \cdots \\ &\sim (i; m-r+1, \dots, m-1, m).\end{aligned}$$

Now we have (see the proof of Lemma 5.4 and use the fact that $[r]_m \neq m$)

$$\begin{aligned}(i; i_1, \dots, i_r) &\sim (i; m-r+1, \dots, m-1, m) \sim ([i+2]_n; 1, m-r+1, \dots, m-1) \\ &\sim ([i+2]_n; m-r+1, \dots, m-1, m) \sim \dots \\ &\sim ([i+2l]_n; m-r+1, \dots, m-1, m) \sim (j; j_1, \dots, j_r).\end{aligned}$$

Conversely, suppose that for all integers l, $[i+2l]_n \neq j$. This can happen only if n is an even number and $[i]_2 \neq [j]_2$. Since the parity of the first coordinate is constant on edges, we see that $(i; i_1, \dots, i_r)$ and $(j; j_1, \dots, j_r)$ cannot be in the same connected component. $\square$

Remark. If $(a_1, a_2, \ldots, a_m)$ is a vertex with r returning points, where $[r]_m = m$, then it is easy to see that for all $i \in m$, $a_{[i+2]_m} \neq a_i$. Hence, there does not exist an edge with this vertex as an endpoint.

6. The Homotopy Type of $\mathtt{Hom}(C_m, C_n)$

Theorem 6.1. *Assume* $\mathtt{Hom}(C_m, C_n) \neq \emptyset$, *and let* X *be some connected component of* $\mathtt{Hom}(C_m, C_n)$. *Then* X *is either a point or is homotopy equivalent to a circle.*

Proof. By Theorem 3.2 we know that the statement is true for $n = 4$, so assume $n \neq 4$.

Lemma 5.3 implies that all vertices in X have the same number of returning points. Denote that number with r. If $r = 0$ or $r = m$, then, clearly, X is a point. We deal with the case when $0 < r < m$.

For all $i \in [n]$, let X_i be the subcomplex of X consisting of closures of all cells η such that $i \in \eta(1)$ and let $\widetilde{X}_i^m$ be the induced subcomplex of X on the vertices η such that $\eta(1) = \{i\}$. It is obvious that $\widetilde{X}_i^m \subseteq X_i$ and that $X = \bigcup_{i=1}^n X_i$. Notice that, in the case when n is even, it is a corollary of Lemma 5.5 that either $X_1 = X_3 = \cdots = X_{n-1} = \emptyset$ or $X_2 = X_4 = \cdots = X_n = \emptyset$.

Claim 1. $\widetilde{X}_i^m$ *is a strong deformation retract of* X_i, *for all* i *such that* $X_i \neq \emptyset$.

Proof of Claim 1. We define a partial matching on $\mathcal{P}(X_i)$ in the following way: for $\eta \in \mathcal{P}(X_i)$ such that $i \notin \eta(1)$, let $\mu(\eta) := \widetilde{\eta}$ where

$$\widetilde{\eta}(j) = \begin{cases} \eta(1) \cup \{i\}, & \text{for} \quad j = 1; \\ \eta(j), & \text{for} \quad j = 2, 3, \ldots, m. \end{cases}$$

Obviously this is an acyclic matching and η is a critical cell if and only if $\eta(1) = \{i\}$. Hence all critical cells form the subcomplex $\widetilde{X}_i^m$. By Proposition 4.3, there exists a sequence of collapses from X_i to $\widetilde{X}_i^m$, and since a collapse is a strong deformation retract, we see that Claim 1 is true and $\widetilde{X}_i^m$ and X_i have the same homotopy type. □

Remark. It is important to see that there *does not* exist an edge between vertices $v = (i; 1, i_2, \ldots, i_r)$ and $w = (i; i_2, \ldots, i_r, m)$ since, in the old notation, $v = (i, i-1, \ldots, i-1)$ and $w = (i, i+1, \ldots, i+1)$ and two vertices of the same edge can be different only on one coordinate. Because of that, when we want to give a picture for a better explanation, instead of C_m we draw L_m. The reason for this notation is that there is no m^+ in notations of cells of $\widetilde{X}_i^m$. For example, the cell $(4; 1, 2, 3^+, 6)$ from $\mathtt{Hom}(C_8, C_8)$ we represent as in Fig. 6.

Claim 2. *If* $\widetilde{X}_i^m \neq \emptyset$, *then it is contractible.*

Proof of Claim 2. Before we define a matching on $\mathcal{P} = \mathcal{P}(\widetilde{X}_i^m)$, we need some additional notations and definitions. For a cell $\eta = (i; s_{i_1}, \ldots, s_{i_r})$ let $F(\eta) = \{i_j \mid j \in$

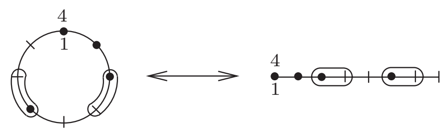

Fig. 6

$[r], i_j \neq m, s_{i_j} = i_j$ and $i_j + 1 \notin \{i_1, \ldots, i_r\}\}$ and $F^+(\eta) = \{i_j \mid j \in [r], s_{i_j} = i_j^+\}$. Also, we define two maps R, R^+: $\widetilde{X}_i^m \to \mathbb{N} \cup \{\infty\}$ in the following way:

For a cell $\eta = (i; s_{i_1}, \ldots, s_{i_r})$ let

$$R(\eta) = \begin{cases} \infty, & F(\eta) = \emptyset, \\ \min F(\eta), & \text{otherwise}, \end{cases}$$

and

$$R^+(\eta) = \begin{cases} \infty, & F^+(\eta) = \emptyset, \\ \min F^+(\eta), & \text{otherwise}. \end{cases}$$

Now, let $S = \{\eta \in \mathcal{P} \mid R(\eta) \lneq R^+(\eta)\}$. In particular, if $\eta \in S$, then $R(\eta) \neq \infty$. For $\eta = (i; s_{i_1}, \ldots, s_{i_r}) \in S$, let $\nu(\eta) = (i; p_{i_1}, \ldots, p_{i_r})$, where

$$p_{i_j} = \begin{cases} s_{i_j}, & i_j \neq R(\eta); \\ i_j^+, & \text{otherwise}. \end{cases}$$

It is clear that ν is injective, $\nu(\eta) \succ \eta$ and $R^+(\nu(\eta)) < R(\nu(\eta))$, hence $\nu(\eta) \notin S$. We conclude that (S, ν) is a partial matching on $\mathcal{P}$. We will now prove that $\sigma = (i; m-r+1, m-r+2, \ldots, m)$ is the only critical cell of our matching. Let $(i; s_{i_1}, \ldots, s_{i_r}) = \xi \in \mathcal{P} \setminus S$. Then we have two cases:

(1) $\infty = R(\xi) = R^+(\xi)$, then $\xi = \sigma$.

(2) $R(\xi) > R^+(\xi)$. In this case let $\xi' = (i; p_{i_1}, \ldots, p_{i_r})$, where

$$p_{i_j} = \begin{cases} s_{i_j}, & i_j \neq R^+(\eta); \\ i_j, & \text{otherwise}. \end{cases}$$

Obviously, $\xi' \in S$, $\nu(\xi') = \xi$, hence ξ is not a critical cell.

We conclude that $\mathcal{P} \setminus (S \cup \nu(S)) = \{\sigma\}$.

What is left to prove is that this matching is acyclic. For any cell $\eta \in \widetilde{X}_i^m$, $\eta = (i; s_{i_1}, \ldots, s_{i_r})$, let $\Sigma(\eta) = \sum_{j=1}^r i_j$. Notice that $\Sigma(\eta) = \Sigma(\nu(\eta))$, for $\eta \in S$.

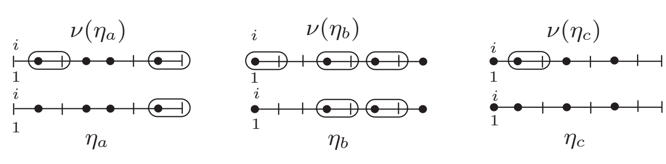

Fig. 7

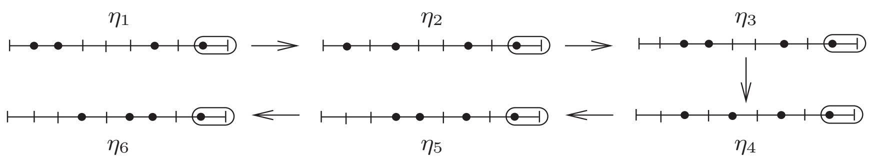

Fig. 8. Example of a sequence $\eta_1, \dots, \eta_6 \in S$ such that $\nu(\eta_i') \succ \eta_{i+1}$, showing how returning points are "moving" to the right.

If $\xi = (i; l, l+1, \dots, l+k, s_{i_{k+2}}, \dots, s_{i_r}) \in S$, where $R(\xi) = l+k$, and if $\xi' \neq \xi$ is a cell such that $\nu(\xi) \succ \xi'$, it is not hard to see that if $\xi' \in S$, then we must have $\xi' = (i; l, l+1, \dots, l+k-1, l+k+1, s_{i_{k+2}}, \dots, s_{i_r})$. Then $\Sigma(\xi') = \Sigma(\xi) + 1$ (here we have also used the previous remark).

Hence, if $\eta_1, \dots, \eta_t \in S$ such that $\eta_1 \neq \eta_2$ and $\nu(\eta_i) \succ \eta_{i+1}$, then $\Sigma(\eta_t) > \Sigma(\eta_1)$ and it is not possible that $\nu(\eta_t) \succ \eta_1$, since in that case it would have to be $\Sigma(\eta_1) = \Sigma(\eta_t)+1$.

By Proposition 4.3 there exists a sequence of elementary collapses leading from $\widetilde{X}_i^m$ to $\{\sigma\}$ and hence $\widetilde{X}_i^m$ is contractible. □

We have proven that all non-empty subcomplexes X_i are contractible. We now determine the structure of their intersections.

Let $\eta = (A_1, A_2, \dots, A_m)$ be a cell such that $\eta \in X_i \cap X_j$, where $i \neq j$. Then, because of the definition of X_i and X_j, we have that $(A_1 \cup \{i\}, A_2, \dots, A_m) \in X_i$ and $(A_1 \cup \{j\}, A_2, \dots, A_m) \in X_j$. However, then, for all $k \in A_2 \cup A_m$, it must be $[k-i]_n, [k-j]_n \in \{1, n-1\}$. This is possible only when $j = [i \pm 2]_n$ (since $n \neq 4$). Without any loss of generality we can assume that $j = [i+2]_n$ and then $A_2 = A_m = \{[i+1]_n\}$. We conclude that

$$X_i \cap X_j = \begin{cases} I \times \widetilde{X}^{m-2}_{[i+1]_n}, & \text{for} \quad j = [i+2]_n \quad \text{and} \quad X_i \neq \emptyset; \\ I \times \widetilde{X}^{m-2}_{[i-1]_n}, & \text{for} \quad j = [i-2]_n \quad \text{and} \quad X_i \neq \emptyset; \\ \emptyset, & \text{otherwise;} \end{cases}$$

where $\widetilde{X}^{m-2}_{[i\pm1]_n}$ are both subcomplexes of $\mathtt{Hom}(C_{m-2}, C_n)$, and I is the unit interval. Notice that each vertex of $\widetilde{X}^{m-2}_{[i+1]_n}$ has $r-1 \geq 0$ returning points. We know that $\widetilde{X}^{m-2}_{[i\pm1]_n}$ are contractible, and hence $X_i \cap X_j$ is also contractible. We also see that $X_{k_1} \cap \dots \cap X_{k_t} = \emptyset$ for $t \geq 3$ (using the similar argument as in case $X_i \cap X_j$).

The family of subcomplexes $\{X_i\}_{i=1}^n$ satisfies the conditions of Corollary 4G.3 and Exercise 4G.4 of [6], and hence we have $X \simeq \mathcal{N}(X_i)$, where $\mathcal{N}(X_i)$ is the nerve of $\{X_i\}_{i=1}^n$. On the other hand, $\mathcal{N}(X_i) = \{i, \{i, [i+2]_n\} \mid i \in [n] \text{ and } X_i \neq \emptyset\} \simeq S^1$ and therefore $X \simeq S^1$. □

Now it is time to summarize our results for complexes $\mathtt{Hom}(C_m, C_n)$. We have several cases depending on the parity of both m and n. First, we specify how we index different connected components. By Lemma 5.5 we have two essentially different cases:

- n is even: We denote with Δ_r^i a connected component in which all vertices have $r \neq 0$, m returning points and for some of its vertices η, $\eta(1) = \{i\}$, where $i \in \{1, 2\}$.

Table 1. $\mathtt{Hom}(C_m, C_n)$.

$n = 4$	$m = 2k$	Homotopy equivalent to S^0
$n = 2l$ $l \neq 2$	$m = 2sl$	$2n$ points and $4s - 2$ connected components which are homotopy equivalent to S^1: $\Delta_l^{1,2}, \Delta_{2l}^{1,2}, \dots, \Delta_{(2s-1)l}^{1,2}$
	$m = 2k$ $l \nmid k$	$2(2[k/l] + 1)$ connected components, all homotopy equivalent to S^1: $\Delta_k^{1,2}, \Delta_{k\pm l}^{1,2}, \dots, \Delta_{k\pm[k/l]l}^{1,2}$
$n = 2l$	$m = 2k + 1$	$\emptyset$
$n = 2l + 1$	$m = sn$	$2n$ points and $s - 1$ connected components $\simeq S^1$: $\Delta_n, \Delta_{2n}, \dots, \Delta_{(s-1)n}$
	$m = 2k + 1$ $n \nmid m$	$[m/n]_{\text{odd}} + 1$ components homotopy equivalent to S^1: $\Delta_{(m\pm n[m/n]_{\text{odd}})/2}, \Delta_{(m\pm n([m/n]_{\text{odd}}-2)/2}, \dots, \Delta_{(m\pm n)/2}$
	$m = 2k$ $n \nmid m$	$[m/n]_{\text{even}} + 1$ connected components, all $\simeq S^1$: $\Delta_k, \Delta_{k\pm n}, \dots, \Delta_{k\pm(n/2)[m/n]_{\text{even}}}$

- n is odd: In this case the only thing which determines a connected component is the number of returning points $r \neq 0,\ m$ of any vertex in that component. Hence, we denote it with Δ_r.

In Table 1 we denote with $[m]$ ($[m]_{\text{odd}}, [m]_{\text{even}}$) the largest integer (resp. the largest odd and the largest even integer) which is less than or equal to m.

Since the Euler characteristic of a point, respectively circle, is equal to 1, respectively 0, we have proven the following claim:

Corollary 6.2.

$$\chi(\mathtt{Hom}(C_m, C_n)) = \begin{cases} 2, & n = 4 \quad \textit{and } m \textit{ is an even number}; \\ 2n, & \textit{when } n \textit{ divides } m \textit{ and } n \neq 4; \\ 0, & \textit{otherwise}. \end{cases}$$

On the other hand, we know that if Y is a cell complex, $\chi(Y) = \sum_n (-1)^n c_n$, where c_n denotes the number of n-cells of Y.

Let $n \neq 4$ and let, as in the proof of Theorem 6.1, X be a connected component of $\mathtt{Hom}(C_m, C_n)$with r returning points, where $\mathtt{Hom}(C_m, C_n) \neq \emptyset$. Then the dimension of a cell from X belongs to the set $\{0, 1, \dots, \min\{r, m-r\}\}$. Let $d \in \{0, 1, \dots, \min\{r, m-r\}\}$. We want to find the explicit formula for the number c_d of d-cells of X.

- If $r = 0$ then $c_0 = 1$ and $c_d = 0$, for $d > 0$.
- Let $r \neq 0$ and $N = n$, if n is odd and $N = n/2$, if n is even. Then $c_0 = N\binom{m}{r}$. Let now $0 < d \leq \min\{r, m - r\}$. We define a map P_d which maps any d-cell

$\eta = (s; s_{i_1}, \dots, s_{i_r})$ from X to a d-tuple of numbers $(i_{j_1}, \dots, i_{j_d})$ such that $j_1 < \dots < j_d$ and $s_{i_{j_k}} = i_{j_k}^{+}$. Also, let S be the set of all d-tuples $(\alpha_1, \dots, \alpha_d)$ which satisfy one of the following two conditions:

$$1 \le \alpha_1 < \dots < \alpha_d < m \quad \text{and} \quad \alpha_{j+1} - \alpha_j \ge 2, \qquad \text{for} \quad j \in [d-1], \tag{6.1}$$

$$1 < \alpha_1 < \dots < \alpha_d = m \quad \text{and} \quad \alpha_{j+1} - \alpha_j \ge 2, \qquad \text{for} \quad j \in [d-1]. \tag{6.2}$$

It is not hard to see that there exists a bijection between the sets $\{P_d(\eta) \mid \eta \in X, \dim(\eta) = d\}$ and S. Since $|P_d^{-1}(s)| = N\binom{m-2d}{r-d}$, for any $s \in S$, we have that $c_d = N\binom{m-2d}{r-d}|S|$. However, the number of d-tuples which satisfy (6.1) is $\binom{m-d}{d}$ and for (6.2) is equal to $\binom{m-d-1}{d-1}$. Hence,

$$c_d = N\left[\binom{m-d}{d} + \binom{m-d-1}{d-1}\right]\binom{m-2d}{r-d} = N\frac{m(m-d-1)!}{d!\,(r-d)!\,(m-r-d)!}.$$

and

$$\chi(X) = \sum_{d=0}^{\min\{r,m-r\}} (-1)^d N \frac{m(m-d-1)!}{d!\,(r-d)!\,(m-r-d)!}.$$

Hence we have proven the following formula, for $1 \le r \le m-1$:

$$\sum_{d=0}^{\min\{r,m-r\}} (-1)^d \frac{(m-d-1)!}{d!\,(r-d)!\,(m-r-d)!}$$
$$= \sum_{d=0}^{\min\{r,m-r\}} (-1)^d d \binom{m-d-1}{d-1, r-d, m-r-d} = 0.$$

7. The Homotopy Type of $\mathrm{Hom}(C_{2m}, L_n)$

It is easy to see that $\mathrm{Hom}(C_m, L_n)$ is empty if m is an odd integer (see the argument for the fact that $\mathrm{Hom}(C_m, C_n)$is empty when n is even and m is odd). Hence, from now on we discuss only the case of $\mathrm{Hom}(C_{2m}, L_n)$. Also, since we have already determined the homotopy type of $\mathrm{Hom}(C_{2m}, L_3)$ and $\mathrm{Hom}(C_{2m}, L_2)$ (see the proof of Theorem 3.2), we assume that $n \ge 4$.

Definition 7.1. We say that $i \in [m]$ is *a returning point* of a vertex $(a_1, a_2, \dots, a_m)$ from $\mathrm{Hom}(C_{2m}, L_n)$ if $a_i - a_{[i+1]_m} = 1$.

Let $v \in \mathrm{Hom}_0(C_{2m}, L_n)$ and let r be the number of its returning points. Then we must have $a_1 + 2m - 2r = a_1$, that is the number of returning points for each vertex must be equal to m.

As we did in the previous section, one can prove that there is a bijection between $\mathrm{Hom}_0(C_{2m}, L_n)$ and the set P, where $P = \{(i; i_1, \dots, i_m) \mid i \in [n], 1 \le i_1 < \dots < i_m \le n$ and $1 \le i + j - 1 - 2|\{q \mid i_q < j\}| \le n,$ for all $j \in [2m]$. Also, we use the same notations for the cells of $\mathrm{Hom}(C_{2m}, L_n)$as we did for the complexes $\mathrm{Hom}(C_m, C_n)$.

Remark. Let $(a_1, a_2, \ldots, a_{2m}) = (i; i_1, \ldots, i_m)$ be a vertex from $\mathrm{Hom}(C_{2m}, L_n)$. If for some l, $[i_l + 1]_{2m} \neq i_{[l+1]_m}$, then $(i; i_1, \ldots, i_l + 1, \ldots, i_m) \in \mathrm{Hom}_0(C_{2m}, L_n)$ (we have only replaced i_l with $i_l + 1$) if and only if $a_{i_l} = i + i_l + 1 - 2l \neq n$.

Lemma 7.2. *Two vertices $u = (i; i_1, \ldots, i_m)$ and $v = (j; j_1, \ldots, j_m)$ of a complex $\mathrm{Hom}(C_{2m}, L_n)$ are in the same connected component if and only if i and j have the same parity.*

Proof. It is easy to see that if $i \neq j \bmod 2$, those vertices cannot be in the same connected component.

Suppose now that i and j have the same parity. Without any loss of generality we can assume that $i \leq j$ and that $j = i + 2q$, for some non-negative integer q. Like in the proof of Lemma 5.5, we define the equivalence relation $\sim$: for two vertices x and y, $x \sim y$ if and only if they lie in the same connected component.

Let now, for all $l \in [n]$, $t^{(l)} = \min\{m, n-l\}$ and let

$$t^{(l)}_{m-k} = \begin{cases} 2m - k, & t^{(l)} \geq 1 \quad \text{and} \quad k \in \{0, 1, \ldots, t^{(l)} - 1\}; \\ 2m - 2k + t^{(l)} - 1, & m \geq t^{(l)} + 1 \quad \text{and} \quad k \in \{t^{(l)}, \ldots, m-1\}. \end{cases}$$

If we have in mind the previous remark, it not hard to see the following:

$$\begin{aligned} (i; i_1, \ldots, i_{r-1}, i_m) &\sim (i; i_1, \ldots, i_{m-1}, t^{(i)}_m) \sim (i; i_1, \ldots, t^{(i)}_{m-1}, t^{(i)}_m) \sim \cdots \\ &\sim (i; t^{(i)}_1, \ldots, t^{(i)}_{m-1}, t^{(i)}_m). \end{aligned}$$

It is now clear that if $i = j$ then $u \sim v$. Suppose that $i < j$. Then, for all $i \leq l < j \leq n$ we have that $t^{(l)}_m = 2m$ and $t^{(l)}_1 \neq 1$. Hence,

$$\begin{aligned} (i; i_1, \ldots, i_m) &\sim (i; t^{(i)}_1, \ldots, t^{(i)}_{m-1}, t^{(i)}_m) \sim (i+2; 1, t^{(i)}_1, \ldots, t^{(i)}_{m-1}) \\ &\sim (i+2; t^{(i+2)}_1, \ldots, t^{(i+2)}_{m-1}, t^{(i+2)}_m) \sim \cdots \\ &\sim (i+2q\,; t^{(i+2q)}_1, \ldots, t^{(i+2q)}_m) \sim (j; j_1, \ldots, j_m). \qquad \square \end{aligned}$$

Theorem 7.3. $\mathrm{Hom}(C_{2m}, L_n)$ *is homotopy equivalent to two points.*

Proof. From Lemma 5.5 we know that $\mathrm{Hom}(C_{2m}, L_n)$has two connected components, namely $\{(s; s_{i_1}, \ldots, s_{i_m}) \in \mathrm{Hom}(C_{2m}, L_n) \mid s \text{ is odd}\}$ and $\{(s; s_{i_1}, \ldots, s_{i_m}) \in \mathrm{Hom}(C_{2m}, L_n) \mid s \text{ is even}\}$. Let X be any of these components. We now prove that X is contractible.

For all $i \in [n]$, define complexes X_i and $\widetilde{X}^m_i$ in the same way as in the proof of Theorem 6.1. We see that $\widetilde{X}^m_i \subseteq X_i$, $X = \bigcup_{i=1}^n X_i$ and $X_2 = X_4 = \cdots = X_{2[n/2]} = \emptyset$ or $X_1 = X_2 = \cdots = X_{2[(n+1)/2]-1} = \emptyset$.

The proof that $\widetilde{X}^m_i$ is a strong deformation retract of X_i, for non-empty X_i, is completely the same as in the proof of Theorem 6.1. For a cell $\eta = (i; s_{i_1}, \ldots, s_{i_r})$ let

$$F(\eta) = \{i_j \mid i_j \neq m, s_{i_j} = i_j, i_j + 1 \notin \{i_1, \ldots, i_r\} \text{ and } i + i_j + 1 - 2j \neq n\}$$

and $F^+(\eta) = \{i_j \mid s_{i_j} = i_j^+\}$. The maps $R, R^+\colon \widetilde{X}_i^m \to \mathbb{N}_0$, the set S and the map ν are also defined analogously to the corresponding objects in the already mentioned proof. By the previous remark, ν is well defined. Again, (S, ν) is a partial matching on $\mathcal{P}(\widetilde{X}_i^m)$. We now prove that $\sigma = (i; t_1^{(i)}, t_2^{(i)}, \dots, t_m^{(i)})$, where $t_1^{(i)}, \dots, t_m^{(i)}$ are defined in the proof of the previous lemma, is the only critical cell of our matching. Let $(i; s_{i_1}, \dots, s_{i_r}) = \xi \in \mathcal{P}\backslash S$. Then we have two cases:

(1) $\infty = R(\xi) = R^+(\xi)$.
We first prove that $\infty = R(\sigma) = R^+(\sigma)$:
- If $k \in \{t^{(i)}, \dots, m-1\}$, then $t^{(i)} = n-i$, $i + t_{m-k}^{(i)} + 1 - 2(m-k) = i + t^{(i)} = n$ and $t_{m-k}^{(i)} \notin F(\sigma)$.
- If $k \in \{0, 1, \dots, t^{(i)} - 1\}$, then it is easy to see that either $t_{m-k+1}^{(i)} = t_{m-k}^{(i)} + 1$ or, for $k = 0$, $t_m^{(i)} = 2m$ and $t_{m-k}^{(i)} \notin F(\sigma)$.

Hence, $F(\sigma) = \emptyset$ and $\infty = R(\sigma) = R^+(\sigma)$.
Since $R^+(\xi) = \infty$, ξ must be a vertex $(i; i_1, \dots, i_m)$.
- First we prove that $i_m = t_m^{(i)}$. Since $R(\xi) = \infty$, $i_m = 2m$ or $i_m < 2m$ and $i + i_m + 1 - 2m = n$. The second case is possible only if $i = n$ and $i_m = 2m-1$. In both cases $i_m = t_m^{(i)}$.
- Suppose now that $i_{m-k} = t_{m-k}^{(i)}$ for some $k \in [m-1]$. We prove that $i_{m-k-1} = t_{m-k-1}^{(i)}$.
  - If $i + t_{m-k}^{(i)} + 1 - 2(m-k) \neq n$, then $k \in \{0, 1, \dots, t^{(i)} - 2\}$ and then we must have $i_{m-k-1} = t_{m-k}^{(i)} - 1 = t_{m-k-1}^{(i)}$.
  - If $i + t_{m-k}^{(i)} + 1 - 2(m-k) = n$, then $k \in \{t^{(i)} - 1, \dots, m-1\}$ and we must have $i_{m-k-1} = t_{m-k}^{(i)} - 2 = t_{m-k-1}^{(i)}$.

Hence we have proven that σ is the only cell with property $\infty = R(\xi) = R^+(\xi)$.

(2) $R^+(\xi) < R(\xi)$:
Let $\xi' = (i; b_{i_1}, \dots, b_{i_r})$, where

$$b_{i_j} = \begin{cases} s_{i_j}, & i_j \neq R^+(\eta); \\ i_j, & \text{otherwise.} \end{cases}$$

Obviously, $\xi' \in S$, $\nu(\xi') = \xi$ and ξ is not a critical cell.

Hence $\mathcal{P}\backslash(S \cup \nu(S)) = \{\sigma\}$.

This matching is acyclic (see again the proof of Lemma 5.5), and hence by Proposition 4.3 there exists a sequence of elementary collapses leading from $\widetilde{X}_i^m$ to $\{\sigma\}$ and hence $\widetilde{X}_i^m$ is contractible.

We have proven that all subcomplexes X_i are contractible.

Let $\eta = (A_1, A_2, \dots, A_m)$ be a cell such that $\eta \in X_i \cap X_j$, where $i < j$. Then $(A_1 \cup \{i\}, A_2, \dots, A_m) \in X_i$ and $(A_1 \cup \{j\}, A_2, \dots, A_m) \in X_j$. However, then, for all $k \in A_2 \cup A_m$, it must be $k - i = \pm 1$, $k - j = \pm 1$. This is possible only when $i \leq n-2$ and $j = i + 2$ and in that case $A_2 = A_m = \{i+1\}$. We conclude that

$$X_i \cap X_j = \begin{cases} I \times \widetilde{X}_{k+1}^{m-2}, & \text{for} \quad |j - i| = 2, \quad k = \min\{i, j\} \quad \text{and} \quad X_i \neq \emptyset, \\ \emptyset, & \text{otherwise,} \end{cases}$$

where $\widetilde{X}_{k+1}^{m-2}$ is a subcomplex of $\mathtt{Hom}(C_{2m-2}, L_n)$, and I is the unit interval. Since $\widetilde{X}_{k+1}^{m-2}$ is contractible, $X_i \cap X_j$ is also contractible. We also see that $X_{i_1} \cap \cdots \cap X_{i_t} = \emptyset$ for $t \geq 3$ (using a similar argument as in the case $X_i \cap X_j$).

The family of subcomplexes $\{X_i\}_{i=1}^n$ satisfies the conditions of Corollary 4G.3 and Exercise 4G.4 of [6] and hence $X \simeq \mathcal{N}(X_i)$. However, in this case $\mathcal{N}(X_i) = \{i \mid X_i \neq \emptyset\} \cup \{\{i, i+2\} \mid X_i \neq \emptyset$ and $i \leq n-2\}$ and, hence, $\mathcal{N}(X_i) \simeq L_k$, where k is the number of non-empty complexes X_i, and L_k is viewed as a one-dimensional simplicial complex. We conclude that X is contractible, and, hence, $\mathtt{Hom}(C_{2m}, L_n)$is homotopy equivalent to two points. □

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